On classes of operators generalizing class A and paranormality and related results (Structure of operators and related current topics)

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On classes of operators generalizing class A and paranormality and related results

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This report is based on the following papers:

[I] M.Ito, *On classes of operators generalizing class A and paranormality*, Sci. Math. Jpn., **57** (2003), 287–297, (online version, **7** (2002), 353–363). (§1–4)

[IYY] M.Ito, T.Yamazaki and M.Yanagida, *Generalizations of results on relations between Furuta-type inequalities*, to appear in Acta Sci. Math. (Szeged). (§5)

Abstract

Recently, we introduced class A defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Nakamoto introduced class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality respectively. These classes are related to p-hyponormality and log-hyponormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $\mathrm{F}(p,r,q)$ operators, and also we shall show that the families of class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality are proper on p. Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the key theorem in the proofs of our main results.

1 Introduction

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that p-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$ and invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator T, and also an operator T is said to be p-quasihyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [26]). It is easily obtained that every p-hyponormal operator is q-hyponormal for

$p > q > 0$ by Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$,"* and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for $X > 0$. An operator T is paranormal if $\|T^2 x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. Ando [2] showed that every p-hyponormal operator for $p > 0$ and (invertible) log-hyponormal operator is paranormal.

Recently, in [15], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms. And also Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [9] introduced class A(p,r) and Yamazaki-Yanagida [28] introduced absolute-(p,r)-paranormality as follows: An operator T belongs to class A(p,r) for $p > 0$ and $r > 0$ if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$, and also an operator T is absolute-(p,r)-paranormal if $\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r}$ for every unit vector $x \in H$. We remark that class A$(1,1)$ equals class A and also absolute-$(1,1)$-paranormality equals paranormality. These classes are generalizations of class A(k) and absolute-k-paranormality introduced as two families of classes based on class A and paranormality in [15], and also absolute-(p,r)-paranormality is a generalization of p-paranormality in [7]. We should remark that the families of class A(p,r) determined by operator inequalities and absolute-(p,r)-paranormality determined by norm inequalities constitute two increasing lines on $p > 0$ and $r > 0$ whose origin is (invertible) log-hyponormality.

Moreover, as a continuation of the discussion in [9], Fujii-Nakamoto [10] introduced the following classes of operators.

Definition ([10]). *For each $p > 0$, $r \geq 0$ and $q > 0$,*

(i) *An operator T belongs to class $F(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.1}$$

(ii) *An operator T is (p,r,q)-paranormal if*

$$\||T|^p U |T|^r x\|^{\frac{1}{q}} \geq \||T|^{\frac{p+r}{q}} x\| \tag{1.2}$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. In particular, if $r > 0$ and $q \geq 1$, then (1.2) *is equivalent to*

$$\||T|^p|T^*|^r x\|^{\frac{1}{q}} \geq \||T^*|^{\frac{p+r}{q}} x\| \tag{1.3}$$

for every unit vector $x \in H$ ([18]).

We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$ and also $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality. In [18], we obtained the parallel result to that of class $A(p, r)$ and absolute-(p, r)-paranormality that invertible class $F(p, r, q)$ and (p, r, q)-paranormality constitute two increasing lines on $p \geq \delta > 0$ and $r \geq r_0 > 0$ whose origin is δ-quasihyponormality. And also we showed the result on powers of invertible class $F(p, r, q)$ operators. Thus many reseachers have been discussed parallel families of classes of operators which are generalizations of class A and paranormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $F(p, r, q)$ operators in [18], and also we shall show that the families of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality are proper on p. Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the result shown in [19] which is the key theorem in the proofs of our main results.

2 Preliminaries

Fujii-Nakamoto [10] observed that class $F(p, r, q)$ derives from the following Theorem 2.A shown in [11] and (p, r, q)-paranormality corresponds to class $F(p, r, q)$.

We remark that alternative proofs of Theorem 2.A were given in [5] and [21] and also an elementary one page proof in [12]. Tanahashi [23] showed that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem 2.A.

Theorem 2.A (Furuta inequality [11]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

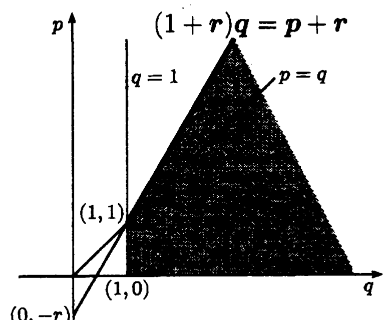

FIGURE 1

Fujii-Nakamoto [10] and the author [18] obtained the results on inclusion relations among the families of class $F(p, r, q)$ and (p, r, q)-paranormality.

Theorem 2.B ([10]).

(i) *For a fixed $k > 0$, T is k-hyponormal if and only if T belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.*

(ii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.*

(iii) *If T is (p_0, r_0, q_0)-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then T is (p_0, r_0, q)-paranormal for any $q \geq q_0$.*

(iv) *If T belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then T is (p, r, q)-paranormal.*

Theorem 2.C ([18]).

(i) *For each $p > 0$ and $r > 0$,*

(i-1) *T is p-quasihyponormal if and only if T belongs to class $F(p, r, 1)$ if and only if T is $(p, r, 1)$-paranormal.*

(i-2) *T is p-quasihyponormal if and only if T is $(p, 0, 1)$-paranormal.*

(ii) *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.*

(ii-1) *If T is invertible and $0 \leq \delta \leq p_0$, then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

(ii-2) *If $-r_0 < \delta \leq p_0$, then T belongs to class $F(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$.*

(iii) *Let T be a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.*

(iii-1) *If $0 \leq \delta \leq p_0$, then T is $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p \geq p_0$ and $r \geq r_0$.*

(iii-2) *If $-r_0 < \delta \leq p_0$, then T is $(p_0, r, \frac{p_0+r}{\delta+r})$-paranormal for any $r \geq r_0$.*

(iii-3) *If $0 \leq \delta$, then T is $(p, r_0, \frac{p+r_0}{\delta+r_0})$-paranormal for any $p \geq p_0$.*

We remark that only (ii-1) of Theorem 2.C requres invertibility of T, and also we obtained in [19] that every class $A(p_0, r_0)$ operator for $p_0 > 0$ and $r_0 > 0$ belongs to class $A(p, r)$ for any $p \geq p_0$ and $r \geq r_0$ (without assumption of invertibility).

Figure 2 on the following page represents the inclusion relations among the families of class $F(p, r, q)$ and (p, r, q)-paranormality.

On the other hand, we obtained the results on powers of p-hyponormal, class $A(p, r)$ and invertible class $F(p, r, q)$ operators.

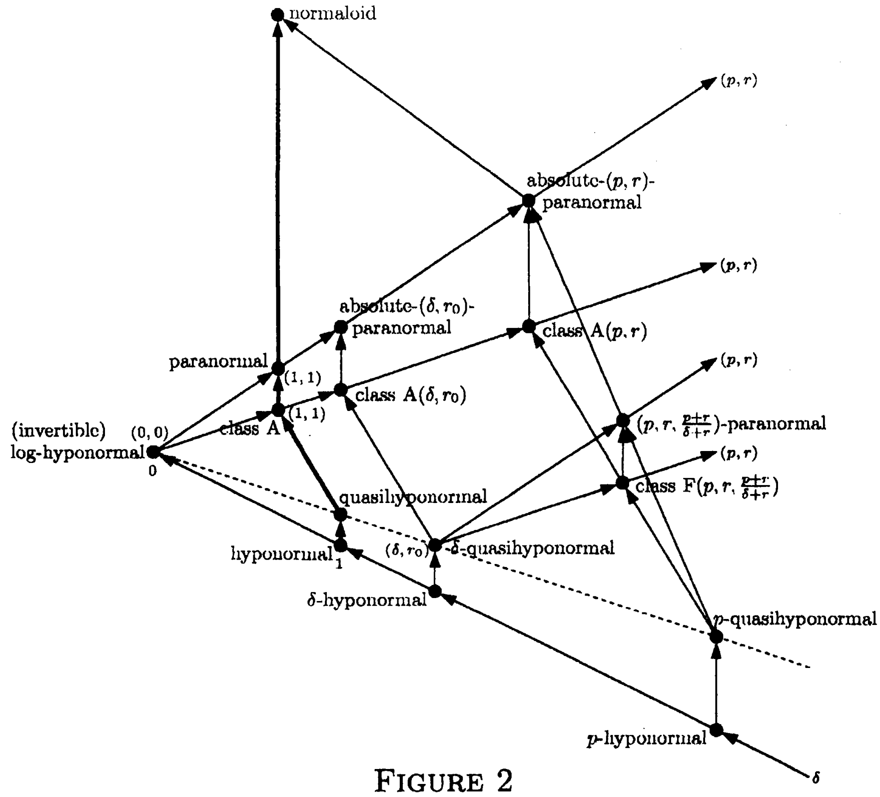

FIGURE 2

Theorem 2.D.

(i) *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then T^n is $\frac{p}{n}$-hyponormal for all positive integer n ([1]).*

(ii) *Let T be a class $A(p,r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then T^n belongs to class $A(\frac{p}{n}, \frac{r}{n})$ for all positive integer n ([19]).*

(iii) *Let T be an invertible class $F(p,r,q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p+r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n ([18]).*

We remark that (iii) interpolates (i) and (ii) if T is invertible in Theorem 2.D. In fact, (iii) yields (i) by putting $q = 1$ and $r = 0$, and also (iii) yields (ii) by putting $q = \frac{p+r}{r}$.

Moreover we have another result on powers of class A operators by combining [29, Theorem 1] and [19, Theorem 3].

Theorem 2.1. *If T is a class A operator, then*

$$|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^{\frac{2}{n}} \quad \textit{and} \quad |T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$$

hold for all positive integer n.

We remark that (ii) of Theorem 2.D and Theorem 2.1 in case of invertible operators were shown in [27] and [17], respectively.

3 Main results

In this section, we shall show the results which remove the assumption of invertibility from (ii-1) of Theorem 2.C and (iii) of Theorem 2.D.

Theorem 3.1. *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$. Then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

Theorem 3.2. *Let T be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n.*

We need the following two results in order to prove Theorem 3.1.

Theorem 3.A ([19, Theorem 1]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

(i) *If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

Theorem 3.B ([29]). *If $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for positive operators A and B and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then*

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}}$$

holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\gamma \geq -\beta_0$,

$$g_{\beta_0,\delta}(\alpha) = (B^{\frac{\beta_0}{2}} A^{\alpha} B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha+\beta_0}}$$

is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence $(B^{\frac{\beta_0}{2}} A^{\alpha_2} B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_2+\beta_0}} \geq B^{\frac{\beta_0}{2}} A^{\alpha_1} B^{\frac{\beta_0}{2}}$ holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Proof of Theorem 3.1. In case $r_0 = 0$, it is already shown in (i) of Theorem 2.B since class $F(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals δ-hyponormality. So we may assume $r_0 > 0$. Suppose that T belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 > 0$ and $0 \leq \delta \leq p_0$, i.e.,

$$(|T^*|^{r_0} |T|^{2p_0} |T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \geq |T^*|^{2(\delta+r_0)}. \tag{3.1}$$

Applying Löwner-Heinz theorem to (3.1), we have

$$(|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{r_0}{p_0+r_0}} \geq |T^*|^{2r_0},$$

and also we have

$$|T|^{2p_0} \geq (|T|^{p_0}|T^*|^{2r_0}|T|^{p_0})^{\frac{p_0}{p_0+r_0}} \tag{3.2}$$

by (i) of Theorem 3.A. By applying Theorem 3.B to (3.2), we obtain that

$$\begin{gathered} g_{r_0,\delta}(p) = (|T^*|^{r_0}|T|^{2p}|T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} \\ \text{is an increasing function for } p \geq \max\{p_0, \delta\} = p_0. \end{gathered} \tag{3.3}$$

Therefore we have

$$\begin{aligned} (|T^*|^{r_0}|T|^{2p}|T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} &= g_{r_0,\delta}(p) \\ &\geq g_{r_0,\delta}(p_0) && \text{by (3.3)} \\ &= (|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \\ &\geq |T^*|^{2(\delta+r_0)} && \text{by (3.1)} \end{aligned}$$

for any $p \geq p_0$, i.e., T belongs to class $\mathrm{F}(p, r_0, \frac{p+r_0}{\delta+r_0})$ for any $p \geq p_0$. Hence T belongs to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$ by (ii-2) of Theorem 2.C. □

To prove Theorem 3.2, we prepare the following result which is a slight modification of [29, Lemma 5].

Lemma 3.3. *Let A, B and C be positive operators, $p > 0$, $0 < r \leq 1$ and $q \geq 1$ with $rq \leq p + r \leq (1+r)q$. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$ and $B \geq C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} \geq C^{\frac{p+r}{q}}$.*

Proof. The hypothesis $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0, 1]$ by Löwner-Heinz theorem. By Douglas' theorem [4], there exists an operator X such that

$$B^{\frac{r}{2}}X = X^*B^{\frac{r}{2}} = C^{\frac{r}{2}} \tag{3.4}$$

and $||X|| \leq 1$. Then we have

$$\begin{aligned} (C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} &= (X^*B^{\frac{r}{2}}A^pB^{\frac{r}{2}}X)^{\frac{1}{q}} \\ &\geq X^*(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}X && \text{by Hansen's inequality [16]} \\ &\geq X^*B^{\frac{p+r}{q}}X && \text{by the hypothesis} \\ &= C^{\frac{r}{2}}B^{\frac{p+r}{q}-r}C^{\frac{r}{2}} && \text{by (3.4) since } \tfrac{p+r}{q} - r \in [0,1] \\ &\geq C^{\frac{p+r}{q}} && \text{by Löwner-Heinz theorem.} \end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 3.2. Let T be a class $\mathrm{F}(p,r,q)$ operator for $0<p\leq 1$, $0\leq r\leq 1$ and $q\geq 1$ with $rq\leq p+r$, i.e.,

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}}\geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.1}$$

Class $\mathrm{F}(p,r,q)$ operator T for $0<p\leq 1$, $0\leq r\leq 1$ and $q\geq 1$ with $rq\leq p+r$ belongs to class $\mathrm{F}(1,1,2)$, i.e., class A by (ii) of Theorem 2.B and Theorem 3.1, and also

$$|T^n|^{\frac{2}{n}}\geq |T|^2 \tag{3.5}$$

and

$$|T^*|^2\geq |T^{n^*}|^{\frac{2}{n}} \tag{3.6}$$

hold for all positive integer n by Theorem 2.1. By applying Lemma 3.3 to (1.1) and (3.6), we have

$$(|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}}\geq |T^{n^*}|^{\frac{2}{n}\frac{p+r}{q}} \tag{3.7}$$

for $0<p\leq 1$, $0\leq r\leq 1$ and $q\geq 1$ with $rq\leq p+r$ since $p+r\leq (1+r)q$ always holds. Hence we obtain

$$\begin{aligned}(|T^{n^*}|^{\frac{r}{n}}|T^n|^{\frac{2p}{n}}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} &\geq (|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} && \text{by (3.5) and Löwner-Heinz theorem}\\ &\geq |T^{n^*}|^{\frac{2}{q}(\frac{p}{n}+\frac{r}{n})} && \text{by (3.7)}\end{aligned}$$

for all positive integer n, that is, T^n belongs to class $\mathrm{F}(\frac{p}{n},\frac{r}{n},q)$ for all positive integer n.
□

4 Properness of class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality

In this section, we shall show the results on inclusion relation among the families of p-quasihyponormality, class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality.

Theorem 4.1. *For each $p_0>0$, there exists a p_0-quasihyponormal operator T such that T is not $(p,r,\frac{p+r}{\delta+r})$-paranormal for any $p>0$, $r>0$ and $\delta>-r$ such that $\delta\leq p<p_0$.*

Theorem 4.2. *For each $p_0>0$, $r_0>0$ and $-r_0<\delta\leq p_0$,*

(i) *There exists a p_0-quasihyponormal operator T such that T is not p-quasihyponormal for any $p>0$ such that $0<p<p_0$.*

(ii) *There exists a class* $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ *operator* T *such that* T *does not belong to class* $F(p, r, \frac{p+r}{\delta+r})$ *for any* $p > 0$ *and* $r > 0$ *such that* $-r < \delta \le p < p_0$.

(iii) *There exists a* $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$*-paranormal operator* T *such that* T *is not* $(p, r, \frac{p+r}{\delta+r})$*-paranormal for any* $p > 0$ *and* $r > 0$ *such that* $-r < \delta \le p < p_0$.

In Theorem 4.2, (i) has been obtained in [24], and also (ii) and (iii) asserts that the families of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality are proper on p. Moreover we remark that these properness on p has no connection with r, and also we have the following corollary by putting $r = r_0$ in Theorem 4.2.

Corollary 4.3. *For each* $p_0 > 0$, $r_0 > 0$ *and* $-r_0 < \delta \le p_0$,

(i) *There exists a class* $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ *operator* T *such that* T *does not belong to class* $F(p, r_0, \frac{p+r_0}{\delta+r_0})$ *for any* $p > 0$ *such that* $\delta \le p < p_0$.

(ii) *There exists a* $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$*-paranormal operator* T *such that* T *is not* $(p, r_0, \frac{p+r_0}{\delta+r_0})$*-paranormal for any* $p > 0$ *such that* $\delta \le p < p_0$.

Here we shall show two propositions as a preparation of the proof of Theorem 4.1. We remark that these propositions are similar arguments to [2], [15], [20] and so on.

Firstly we shall give a characterization of (p, r, q)-paranormal operators.

Proposition 4.4. *For each* $p > 0$, $r > 0$ *and* $-r < \delta \le p$, *an operator* T *is* $(p, r, \frac{p+r}{\delta+r})$*-paranormal if and only if*

$$(\delta + r)|T^*|^r|T|^{2p}|T^*|^r - (p + r)\lambda^{p-\delta}|T^*|^{2(\delta+r)} + (p - \delta)\lambda^{p+r} \ge 0 \quad \textit{for all } \lambda > 0.$$

Proof. Suppose that T is $(p, r, \frac{p+r}{\delta+r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \le p$, i.e.,

$$\big\||T|^p|T^*|^r x\big\|^{\frac{\delta+r}{p+r}} \ge \big\||T^*|^{\delta+r}x\big\| \quad \text{for every unit vector } x \in H. \tag{1.3}$$

(1.3) holds iff

$$\big\||T|^p|T^*|^r x\big\|^{\frac{\delta+r}{p+r}}\big\|x\big\|^{\frac{p-\delta}{p+r}} \ge \big\||T^*|^{\delta+r}x\big\| \quad \text{for all } x \in H$$

iff

$$(|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{\delta+r}{p+r}}(x, x)^{\frac{p-\delta}{p+r}} \ge (|T^*|^{2(\delta+r)}x, x) \quad \text{for all } x \in H. \tag{4.1}$$

By arithmetic-geometric mean inequality,

$$
\begin{aligned}
&(|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{\delta+r}{p+r}}(x,x)^{\frac{p-\delta}{p+r}} \\
&= \left\{\left(\frac{1}{\lambda}\right)^{p-\delta}(|T^*|^r|T|^{2p}|T^*|^r x, x)\right\}^{\frac{\delta+r}{p+r}} \cdot \{\lambda^{\delta+r}(x,x)\}^{\frac{p-\delta}{p+r}} \\
&\leq \frac{\delta+r}{p+r}\frac{1}{\lambda^{p-\delta}}(|T^*|^r|T|^{2p}|T^*|^r x, x) + \frac{p-\delta}{p+r}\lambda^{\delta+r}(x,x)
\end{aligned}
\tag{4.2}
$$

for all $x \in H$ and all $\lambda > 0$, so (4.1) ensures the following (4.3) by (4.2).

$$
\frac{\delta+r}{p+r}\frac{1}{\lambda^{p-\delta}}(|T^*|^r|T|^{2p}|T^*|^r x, x) + \frac{p-\delta}{p+r}\lambda^{\delta+r}(x,x) \geq (|T^*|^{2(\delta+r)}x, x) \quad \text{for all } x \in H \text{ and all } \lambda > 0. \tag{4.3}
$$

Conversely, (4.1) follows from (4.3) by putting $\lambda = \left\{\frac{(|T^*|^r|T|^{2p}|T^*|^r x, x)}{(x,x)}\right\}^{\frac{1}{p+r}}$. (In case $(|T^*|^r|T|^{2p}|T^*|^r x, x) = 0$, let $\lambda \to +0$.) Hence (4.3) holds if and only if

$$
(\delta+r)|T^*|^r|T|^{2p}|T^*|^r - (p+r)\lambda^{p-\delta}|T^*|^{2(\delta+r)} + (p-\delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0,
$$

so that the proof is complete. □

Secondly we shall give the following Proposition 4.5. But we omit to describe these calculation because it is obtained by easy calculation.

Proposition 4.5. *Let* $K = \bigoplus_{n=-\infty}^{\infty} H_n$ *where* $H_n \cong H$. *For given positive operators* A, B *on* H, *define the operator* $T_{A,B}$ *on* K *as follows:*

$$
T_{A,B} = \begin{pmatrix}
\ddots & & & & & & \\
\ddots & 0 & & & & & \\
& B^{\frac{1}{2}} & 0 & & & & \\
& & B^{\frac{1}{2}} & \boxed{0} & & & \\
& & & A^{\frac{1}{2}} & 0 & & \\
& & & & A^{\frac{1}{2}} & 0 & \\
& & & & & \ddots & \ddots
\end{pmatrix}, \tag{4.4}
$$

where $\square$ *shows the place of the* $(0,0)$ *matrix element.*

(i) *For each* $p > 0$, $T_{A,B}$ *is p-quasihyponormal if and only if*

$$
B^{\frac{1}{2}}A^pB^{\frac{1}{2}} \geq B^{p+1}.
$$

(ii) *For each* $p > 0$, $r \geq 0$ *and* $\delta \geq -r$, $T_{A,B}$ *belongs to class* $F(p, r, \frac{p+r}{\delta+r})$ *if and only if*

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}.$$

(iii) *For each* $p > 0$, $r > 0$ *and* $-r < \delta \leq p$, $T_{A,B}$ *is* $(p, r, \frac{p+r}{\delta+r})$*-paranormal if and only if*

$$(\delta + r)B^{\frac{r}{2}} A^p B^{\frac{r}{2}} - (p + r)\lambda^{p-\delta} B^{\delta+r} + (p - \delta)\lambda^{p+r} I \geq 0 \quad \textit{for all } \lambda > 0.$$

Proof of Theorem 4.1. Let

$$\begin{aligned} &A = U\Lambda U^* \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad \text{where } U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} (2 - e^{-p_0})^{\frac{1}{p_0}} & 0 \\ 0 & e^{-2} \end{pmatrix}, \end{aligned} \tag{4.5}$$

and also let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong \mathbb{R}^2$. For positive matrices A, B on $\mathbb{R}^2$ given in (4.5), define the operator $T_{A,B}$ on K as (4.4) in Proposition 4.5. By (i) of Proposition 4.5, $T_{A,B}$ is p-quasihyponormal for $p > 0$ if and only if

$$B^{\frac{1}{2}} A^p B^{\frac{1}{2}} - B^{p+1} = \begin{pmatrix} \frac{1}{2}\{(2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p}\} - 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

if and only if

$$f(p) \equiv \frac{1}{2}\{(2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p}\} - 1 \geq 0.$$

On the other hand, let $X_p(\lambda)$ as

$$\begin{aligned} X_p(\lambda) &\equiv (\delta + r)B^{\frac{r}{2}} A^p B^{\frac{r}{2}} - (p + r)\lambda^{p-\delta} B^{\delta+r} + (p - \delta)\lambda^{p+r} I \\ &= \begin{pmatrix} \frac{1}{2}(\delta + r)\{(2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p}\} - (p + r)\lambda^{p-\delta} + (p - \delta)\lambda^{p+r} & 0 \\ 0 & (p - \delta)\lambda^{p+r} \end{pmatrix}. \end{aligned}$$

By (iii) of Proposition 4.5, $T_{A,B}$ is $(p, r, \frac{p+r}{\delta+r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$ if and only if $X_p(\lambda) \geq 0$ for all $\lambda > 0$ if and only if

$$g_p(\lambda) \equiv \frac{1}{2}(\delta + r)\{(2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p}\} - (p + r)\lambda^{p-\delta} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0 \tag{4.6}$$

since $(p - \delta)\lambda^{p+r} \geq 0$ for all $\lambda > 0$. Since $g_p'(\lambda) = (p + r)(p - \delta)\lambda^{p-\delta-1}(-1 + \lambda^{\delta+r})$, we get that

$$\min_{\lambda > 0} g_p(\lambda) = g_p(1) = \frac{1}{2}(\delta + r)\{(2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p}\} - (\delta + r) = (\delta + r)f(p),$$

so that (4.6) holds if and only if $f(p) \geq 0$.

$f(p)$ is a convex function for $p > 0$ since

$$f''(p) = \frac{1}{2}[(2 - e^{-p_0})^{\frac{p}{p_0}}\{\log(2 - e^{-p_0})^{\frac{1}{p_0}}\}^2 + 4e^{-2p}] > 0 \quad \text{for all } p > 0,$$

and also $f(p) = 0$ if $p = 0, p_0$. So we have $f(p_0) = 0$ but $f(p) < 0$ for $0 < p < p_0$. Therefore $g_p(1) < 0$, that is $X_p(1) \not\geq 0$ for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.

Hence $T_{A,B}$ is p_0-quasihyponormal but non-$(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$, so the proof is complete. □

Proof of Theorem 4.2. Let $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$.

Proof of (i). By (i-1) of Theorem 2.C, T is p-quasihyponormal if and only if T is $(p, r, 1)$-paranormal for some $p > 0$ and $r > 0$. Therefore there exists a p_0-quasihyponormal operator T such that T is not p-quasihyponormal for any $0 < p < p_0$ by putting $\delta = p$ in Theorem 4.1.

Proof of (ii). By (i-1) of Theorem 2.C and (ii) of Theorem 2.B, every p_0-quasihyponormal operator belongs to class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$. And also, by (iv) of Theorem 2.B, T does not belong to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ if T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for each $p > 0$, $r > 0$ and $-r < \delta \leq p$. Therefore there exists a class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 4.1.

Proof of (iii). By (i-1) of Theorem 2.C and (iii) of Theorem 2.B, every p_0-quasihyponormal operator is $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal. Therefore there exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 4.1. □

Remark 1. In [15], we introduced two families of classes of operators based on class A and paranormality as follows: An operator T belongs to class $\mathrm{A}(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, and also an operator T is absolute-k-paranormal for $k > 0$ if $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in H$. In [7], Fujii-Izumino-Nakamoto introduced p-paranormality for $p > 0$ defined by $\||T|^p U|T|^p x\| \geq \||T|^p x\|^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. It was pointed out in [27] that class $\mathrm{A}(k)$ equals class $\mathrm{A}(k, 1)$, and also it was shown in [28] that absolute-k-paranormality equals absolute-$(k, 1)$-paranormality and p-paranormality equals absolute-(p, p)-paranormality. We ramark that p-paranormality corresponds to class $\mathrm{A}(p, p)$. We shall also get the results on inclusion relation among the families of these classes.

Corollary 4.6.

(i) *For each $k_0 > 0$, there exists a class $A(k_0)$ operator T such that T does not belong to class $A(k)$ for any $0 < k < k_0$.*

(ii) *For each $k_0 > 0$, there exists an absolute-k_0-paranormal operator T such that T is not absolute-k-paranormal for any $0 < k < k_0$.*

(iii) *For each $p_0 > 0$, there exists a class $A(p_0, p_0)$ operator T such that T is not class $A(p,p)$ for any $0 < p < p_0$.*

(iv) *For each $p_0 > 0$, there exists a p_0-paranormal operator T such that T is not p-paranormal for any $0 < p < p_0$.*

Proof of Corollary 4.6.
Proofs of (i) *and* (ii). By putting $p_0 = k_0$, $r_0 = 1$, $\delta = 0$ and $p = k$ in Corollary 4.3, we have (i) and (ii) since class A(k) equals class F$(k, 1, k+1)$ and absolute-k-paranormality equals $(k, 1, k+1)$-paranormality.

Proofs of (iii) *and* (iv). By putting $p_0 = r_0$, $\delta = 0$ and $p = r$ in (ii) and (iii) of Theorem 4.2, we have (iii) and (iv) since class A(p,p) equals class F$(p,p,2)$ and p-paranormality equals $(p,p,2)$-paranormality. □

Remark 2. For each $p > 0$, we can obtain an example of non-class A(p,p) and p-paranormal operators by using essentially the same example as [15, (2) of Example 8] as follows: Let $p > 0$ and

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix}^{\frac{2}{p}} \quad \text{and} \quad B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}^{\frac{2}{p}}.$$

Then

$$(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} - B^p = \begin{pmatrix} 0.17472\ldots & -3.1798\ldots \\ -3.1798\ldots & 11.770\ldots \end{pmatrix}.$$

Eigenvalues of $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} - B^p$ are $12.585\ldots$ and $-0.64001\ldots$, so that $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} \not\geq B^p$. So $T_{A,B}$ is a non-class A(p,p) operator by (ii) of Proposition 4.5.

On the other hand, for $\lambda > 0$, define $X(\lambda)$ as follows:

$$X(\lambda) \equiv B^{\frac{p}{2}} A^p B^{\frac{p}{2}} - 2\lambda B^p + \lambda^2 I = \begin{pmatrix} 404 - 26\lambda + \lambda^2 & -576 + 24\lambda \\ -576 + 24\lambda & 844 - 26\lambda + \lambda^2 \end{pmatrix}.$$

Put $p(\lambda) = \operatorname{tr} X(\lambda)$ and $q(\lambda) = \det X(\lambda)$, where $\operatorname{tr} X$ denotes the trace of a matrix X and $\det X$ denotes the determinant of a matrix X. Then

$$\begin{aligned} p(\lambda) &= 2\lambda^2 - 52\lambda + 1248 \\ &= 2(\lambda - 13)^2 + 910 > 0 \end{aligned}$$

$$
\begin{aligned}
q(\lambda) &= (404 - 26\lambda + \lambda^2)(844 - 26\lambda + \lambda^2) - (-576 + 24\lambda)^2 \\
&= \lambda^4 - 52\lambda^3 + 1348\lambda^2 - 4800\lambda + 9200.
\end{aligned}
$$

By calculation,

$$
\begin{aligned}
q'(\lambda) &= 4\lambda^3 - 156\lambda^2 + 2696\lambda - 4800 \\
&= 4(\lambda - 2)(\lambda^2 - 37\lambda + 600) \\
&= 4(\lambda - 2)\left\{\left(\lambda - \frac{37}{2}\right)^2 + \frac{1031}{4}\right\}.
\end{aligned}
$$

So $q'(\lambda) = 0$ iff $\lambda = 2$, that is, $q(\lambda) \geq q(2) = 4592 > 0$ for all $\lambda > 0$. Hence $X(\lambda) \geq 0$ for all $\lambda > 0$ since $\operatorname{tr} X(\lambda) = p(\lambda) > 0$ and $\det X(\lambda) = q(\lambda) > 0$ for all $\lambda > 0$. Therefore $T_{A,B}$ is a p-paranormal operator since $T_{A,B}$ is p-paranormal if and only if

$$
pB^{\frac{p}{2}}A^pB^{\frac{p}{2}} - 2p\mu^pB^p + p\mu^{2p}I \geq 0 \quad \text{for all } \mu > 0
$$

if and only if

$$
B^{\frac{p}{2}}A^pB^{\frac{p}{2}} - 2\lambda B^p + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0.
$$

by (iii) of Proposition 4.5.

5 Relations between Furuta-type inequalities

In this section, we shall show a generalization of Theorem 3.A which plays an important role in the proofs of the results in Section 3. Here we recall Theorem 3.A.

Theorem 3.A ([19, Theorem 1]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

(i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

For positive invertible operators A and B, it was shown in [13] that

$$
(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Longleftrightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \tag{5.1}
$$

for fixed positive numbers $p \geq 0$ and $r \geq 0$, and Theorem 3.A is a general result for a relation between two inequalities in (5.1). We remark that it was shown in [6] and [13] (see also [3][8][25]) as an application of Theorem F that for positive invertible operators A and B,

$$\begin{aligned} \log A \geq \log B &\iff (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ for all } p \geq 0 \text{ and } r \geq 0, \\ &\iff A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq 0. \end{aligned} \tag{5.2}$$

As an extension of (5.2) and an immediate corollary of results on operator-valued functions in [6] and [13], we have that for positive invertible operators A and B,

$$\begin{aligned} \log A \geq \log B &\iff (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}} \text{ for all } p \geq \gamma \geq 0 \text{ and } r \geq 0, \\ &\iff A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\delta+p}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq \delta \geq 0. \end{aligned} \tag{5.3}$$

We remark that inequalities of type of (5.3) were initiated in [21].

Here we shall show a generalization of Theorem 3.A on inequalities in (5.3).

Theorem 5.1. *Let A and B be positive operators. Then the following assertions hold, where S^0 means the projection onto $N(S)^{\perp}$ for a positive operator S:*

(i) *For each $r \geq \delta \geq 0$ and $p \geq 0$,*

(i-1) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta}$ *ensures* $A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$,

(i-2) $A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$ *and* $N(AB^{\frac{\delta}{2}}) = N(B)$ *ensure* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta}$.

(ii) *For each $p \geq \gamma \geq 0$ and $r \geq 0$,* $A^{p-\gamma} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}$ *is equivalent to* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}$.

We remark that two inequalities in (i) and (ii) of Theorem 5.1 are mutually equivalent in case A and B are both invertible [22].

We use the following lemma in order to give a proof of Theorem 5.1. Throughout this section, $P_{\mathcal{M}}$ denotes the projection onto a closed subspace $\mathcal{M}$, and also $S^0 = P_{N(S)^{\perp}}$ for a positive operator S.

Lemma 5.2. *Let A and B be positive operators. Then the following assertions hold:*

(i) $\lim\limits_{\varepsilon\to+0} A^{\frac{1}{2}}(A+\varepsilon I)^{-1}A^{\frac{1}{2}} = \lim\limits_{\varepsilon\to+0}(A+\varepsilon I)^{-1}A = P_{N(A)^{\perp}}$.

(ii) $\lim\limits_{\varepsilon\to+0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha}+\varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$ *for* $\alpha \in (0,1]$.
Particularly, in case $\alpha = 1$,
$\lim\limits_{\varepsilon\to+0} A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}}+\varepsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^{\perp}}$.

For positive invertible operators A and B, equivalence between two inequalities in (i) or (ii) of Theorem 5.1 can be easily proved by applying the following Lemma 5.A.

Lemma 5.A ([14]). *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

We remark that for non-invertible operators A and B, Lemma 5.A is valid in case $\lambda \geq 1$ but cannot be applied in case $\lambda \in [0,1)$. For positive invertible operators A and B, Lemma 5.A can be rewritten as

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$$

for any real number α, so that we can regard (ii) of Lemma 5.2 as a non-invertible version of Lemma 5.A for $\alpha \in (0,1]$.

Proof of Lemma 5.2. (i) is well known and a proof was given in [19], for example.
Proof of (ii). Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition. For $\alpha \in (0,1]$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to+0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^\alpha + \varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \\
&= \lim_{\varepsilon\to+0} U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\alpha}(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2\alpha}+\varepsilon I)^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^* \\
&= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}P_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)^\perp}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^* \qquad \text{by (i)} \\
&= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2(1-\alpha)}U^* = |B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2(1-\alpha)} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}.
\end{aligned}$$

We remark that in case $\alpha = 1$ particularly,

$$U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^0U^* = UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)^\perp}U^* = UU^*UU^* = UU^* = P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^0.$$

Hence the proof is complete. □

Proof of Theorem 5.1.
Proof of (i). Let $r > \delta \geq 0$ since the case $r = \delta$ is obvious. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta}$, then

$$A^{\frac{p}{2}}B^{\frac{\delta}{2}}B^{\frac{r-\delta}{2}}(B^{r-\delta}+\varepsilon I)^{-1}B^{\frac{r-\delta}{2}}B^{\frac{\delta}{2}}A^{\frac{p}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}}+\varepsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}$$

for $\varepsilon > 0$, so that

$$A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}} = A^{\frac{p}{2}}B^{\frac{\delta}{2}}P_{N(B)^\perp}B^{\frac{\delta}{2}}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$$

by tending $\varepsilon \to +0$ and Lemma 5.2, hence we obtain (i-1). On the other hand, if $A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$, then

$$B^{\frac{r}{2}}A^{\frac{p}{2}}\{(A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}+\varepsilon I\}^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}} \geq B^{\frac{r-\delta}{2}}B^{\frac{\delta}{2}}A^{\frac{p}{2}}(A^{\frac{p}{2}}B^{\delta}A^{\frac{p}{2}}+\varepsilon I)^{-1}A^{\frac{p}{2}}B^{\frac{\delta}{2}}B^{\frac{r-\delta}{2}}$$

for $\varepsilon > 0$, so that

$$\begin{aligned}(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} &\geq B^{\frac{r-\delta}{2}}P_{N(A^{\frac{p}{2}}B^{\frac{\delta}{2}})^{\perp}}B^{\frac{r-\delta}{2}} && \text{by tending } \varepsilon \to +0 \text{ and (ii) of Lemma 5.2}\\ &= B^{\frac{r-\delta}{2}}P_{N(B)^{\perp}}B^{\frac{r-\delta}{2}} && \text{by } N(AB^{\frac{\delta}{2}}) = N(B)\\ &= B^{r-\delta},\end{aligned}$$

hence we obtain (i-2).

Proof of (ii). Let $p > \gamma \geq 0$ since the case $p = \gamma$ is obvious. If $A^{p-\gamma} \geq (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}$, then

$$B^{\frac{r}{2}}A^{\frac{p}{2}}\{(A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}+\varepsilon I\}^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}} \geq B^{\frac{r}{2}}A^{\frac{\gamma}{2}}A^{\frac{p-\gamma}{2}}(A^{p-\gamma}+\varepsilon I)^{-1}A^{\frac{p-\gamma}{2}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}}$$

for $\varepsilon > 0$, so that

$$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\frac{\gamma}{2}}P_{N(A)^{\perp}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}} = B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}$$

by tending $\varepsilon \to +0$ and Lemma 5.2, hence we obtain ($\Longrightarrow$). On the other hand, if $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}$, then

$$A^{\frac{p-\gamma}{2}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}}(B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}+\varepsilon I)^{-1}B^{\frac{r}{2}}A^{\frac{\gamma}{2}}A^{\frac{p-\gamma}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}}+\varepsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}$$

for $\varepsilon > 0$, so that

$$A^{p-\gamma} \geq A^{\frac{p-\gamma}{2}}P_{N(B^{\frac{r}{2}}A^{\frac{\gamma}{2}})^{\perp}}A^{\frac{p-\gamma}{2}} \geq (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}$$

by tending $\varepsilon \to +0$ and (ii) of Lemma 5.2, hence we obtain ($\Longleftarrow$). □

Theorem 3.A can be obtained as a corollary of Theorem 5.1 as follows.

Alternative proof of Theorem 3.A. Put $\delta = 0$ in (i-1) of Theorem 5.1, then $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$ ensures

$$A^{p} \geq A^{\frac{p}{2}}P_{N(B)^{\perp}}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{p}{p+r}},$$

hence we obtain (i). On the other hand, put $\gamma = 0$ in (ii) of Theorem 5.1, then $A^{p} \geq (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}})^{\frac{p}{p+r}}$ ensures

$$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{\frac{r}{2}}P_{N(A)^{\perp}}B^{\frac{r}{2}} \geq B^{\frac{r}{2}}P_{N(B)^{\perp}}B^{\frac{r}{2}} = B^{r}$$

since $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)^{\perp}} \geq P_{N(B)^{\perp}}$, hence we obtain (ii). □

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